Back to the Sixties: A Note on Multi-Primary-Factor Linear Models with Homogeneous Capital

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Abstract

This paper extends Bruno's (1967) one capital good two-sector growth model with discrete technology by allowing for multiple primary factors of production. While the existence of an optimal steady state is established for any positive rate of discount, an example in which three "modified golden rules" exist shows that the optimal steady state is not necessarily unique. The extended model provides a simple exemplification of the more general principle that the presence of multiple primary factors of production in homogeneous capital models can definitively result in the same complications that arise when there is joint production.

Keywords: Homogeneous capital, Multiple primary factors, Linear activity models, Duality.

JEL Classification: C62, O41.

1 Introduction

A recently published unfinished handwritten manuscript by Paul Samuelson (transcribed by Edwin Burmeister) (Samuelson & Burmeister, 2016) outlines a linear activity model with alternative known techniques each involving, along with labour and corn seed, non-reproducible land and then allows the possibility that each category of inputs involves heterogeneous varieties. The manuscript is a highly incomplete first draft that comprises an introduction, outlining the model and the plan of the paper, and a few lines of a section in which an example of a smooth neoclassical production function with two primary factors (land and labour) and two capital goods is presented. The model is introduced as belonging to "the non-Clark Sraffa-Samuelson paradigm", which is contrasted

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with "the Clark paradigm" where technology is smooth. Samuelson's plan was to begin with the short-period problem of competitive pricing given amounts of the different factors of production and then to consider "the intertemporal phasing of technology" (Samuelson & Burmeister, 2016, p. 160) and hence the fact that "the produced inputs currently used [*are*] themselves the outputs of earlier periods" (Samuelson & Burmeister, 2016, p. 161). The final aim was "to explicate objectively what reswitching and all that implies for the pretensions of Clarkian smooth technologies and the pretensions of Sraffa-Samuelson discrete technique marginalisms" (Samuelson & Burmeister, 2016, p. 162).

With regard to the short-run competitive factor pricing for the discrete linear activity case, Samuelson's conclusion is that it can be reduced to the solution of a pair of dual linear programming problems, with linear programming results providing for this case the "different "Sraffa-Samuelson marginalism"" (Samuelson & Burmeister, 2016, p. 162) that is needed to determine the income distribution. The main conclusion regarding long-run comparative statics is stated instead in the last paragraph of the introduction[1]:

> For readers on the run, I will merely state at this point that solely two classical and neoclassical tenets are shown to be in need of careful justification: in the absence of technical innovations, as the interest rate falls from one steady state to another, as soon as there are joint products and/or multiple heterogeneous capital goods, there need not be an induced rise in the plateau of permanently consumable output. None of the other of my numerically stated points listed above turn out to be negated. Furthermore, qualitatively, all that gets ruled out in either the Clark paradigm or in the non-Clark Sraffa-Samuelson paradigm, must also get ruled out in both of these non-identical paradigms.
>
> Samuelson & Burmeister (2016, p. 162-3)

Read in connection with various passages in two companion papers (Samuelson & Etula, 2006 and Samuelson, 2007), where a "qualitative parallelism of non-spurious marginalisms for Clark and Sraffa technologies" is proved (see, for example, Figure 2 on p. 339 of Samuelson & Etula, 2006), this passage suggests that at some point in time Samuelson thought of a third conclusion that can be drawn "for the case of three factors– n = 3, corresponding say to homogeneous one-quality land, A , homogeneous one-quality labor, L , and homogeneous one-quality corn seed, K" (Samuelson & Burmeister, 2016, p. 161), and more generally for all cases in which capital is homogeneous: namely, appropriately

[1] As pointed out by a referee, the first-draft nature of this paragraph is evident and almost surely Samuelson would have revised it once his work was completed. So it is not surprising that some slips or ambiguities appear. As an example, he suggested that more than one interpretation is possible for the quoted passage as with the expression "two classical and neoclassical tenets" Samuelson could have referred either to two monotonic relationships holding in basic classical and neoclassical models, of which just one (the rise of steady state consumption as the interest rate falls) is then mentioned, or to the absence of joint production and multiple capital goods.

reinterpreted to take into account that in the discrete case the "production functions" are neither differentiable nor strictly concave, the "neoclassical" comparative statics properties of the Solow-Ramsey model hold if joint production is excluded. A remark explaining the simple way in which the capital rental rate is linked to the interest rate in the case of three factors proves that Samuelson was in effect thinking of a multi-factor one final good model, where, as we will see, the conclusion is granted:

> Remark: When K is a produced input like corn seed, which is totally used up in one period's use (which is the case for Sraffa's 1960 Part I), we avoid all joint-production complications. In this case $R_K - 1$ is the "own corn rate of interest." However, if K were a "durable machine" that in each use depreciates by say 10%, then $R_K - 0.1$ would be the own rate of interest in any stationary state. The reader and I have no need in the present exposition to concern ourselves with profit or interest rates so long as we do keep in focus K's gross rental rate R_K^*.
>
> Samuelson & Burmeister (2016, p. 161-2)

The aim of this note is to assess whether the "neoclassical" properties survive in two-sector homogeneous-capital linear models with a single consumption good *à la* Bruno (1967). Our main result is that they do, as expected, when the two sectors have the same technology, and so the system collapses to a one-commodity system, but fails in the general case, where "Wicksell price effects" sever the simple linkage between the rental rate of capital and the capital good's "own interest rate" which is described in the previous remark. To prove this point, we provide a specific example where a finite number of multiple steady states exists even if proper joint production is excluded.[2] In a sense, in homogeneous capital models the multiplicity of primary factors of production acts as a substitute for joint production in allowing multiple turnpikes as in Liviatan & Samuelson (1969) or Burmeister & Turnovsky (1972). Note that a slightly different interpretation was advanced in Burmeister (1975), where it was suggested that "a kind of joint intrinsic production [...] occurs when the number of primary factors exceeds one" (Burmeister, 1975, p. 500).

In the construction of the example showing that multiple steady state can exist in the classical one capital good linear activity model developed by Bruno (1967), provided at least two primary factors (two qualities of labour, for example) are required in the production of the two goods of the system,[3] we build on similar results that can be obtained in one capital good Ricardian models with intensive rent (Freni, 1991, 1997).

[2]Note that multiple steady states in the form of a continuum of turnpikes belonging to a convex set occur in all kinds of linear model whenever the stationarity conditions are satisfied at a switch point.

[3]A multi-sector version of the model without primary factors of production and with heterogeneous capital goods and a CRRA utility function has been studied in the endogenous growth literature see e.g. Freni *et al.* (2003, 2006, 2008).

The continuous time framework used here precludes a direct comparison with the discrete time case of circulating capital and no attempt is made in this work to establish whether the discrete time case with non-durable capital enjoys specific properties.[4] Strictly speaking indeed, fixed capital cannot be avoided in continuous time. In continuous time, however, joint production occurs if the flow output vector of at least a process contains more than one positive entry and this is not implied by fixed capital as such, as it is instead in discrete time. In particular, since we stick with the usual assumption that the rate of depreciation of capital is a constant not affected by capital utilization, our scenario is one in which there is single production despite the fact that capital is durable. Hence, what the example shows is that the presence of heterogeneous primary factors in the classical one-capital two-sector growth model can definitively result in the complications that arise when there is joint production (cfr. Etula, 2008, p. 100).

A multiple-primary-factor extension of Bruno's (1967) two-sector model is briefly reviewed in Section 2. The example is presented in Section 3. Section 4 concludes.

2 A two-sector multiple-primary-factor linear model

Consider the two-sector multiple-technique case of the discrete capital model introduced in Bruno (1967) under the hypothesis that multiple primary factors in fixed supply are used in production. In the system, there are two commodities: a pure capital good and a pure consumption good. The services of s, $s \geq 1$, primary factors of production, different qualities of labour for simplicity, are combined with the services of the stock of capital to produce the two commodities. Technology is of the discrete type without joint production, comprising m, $m \geq 1$, processes for producing the consumption good and n, $n \geq 1$, processes that produce the capital good.

When process j, $j \in \{1, 2, ..., n\}$, is used, a unit of the capital good needs, to be produced, a_{kj} units of the capital good services and $[l_{kj1}, l_{kj2}, ..., l_{kjs}]$ units of the services of the primary factors of production, whereas the production of one unit of the consumption good by means of the i-th, $i \in \{1, 2, ..., m\}$, process requires a_{ci} units of the capital good services and $[l_{ci1}, l_{ci2}, ..., l_{cis}]$ units of the services of the primary factors of production. Hence the technology is described by a couple of capital coefficients vectors

$$\mathbf{a}_c = \begin{bmatrix} a_{c1} & a_{c2} & ... & a_{cm} \end{bmatrix}^T, \quad \mathbf{a}_k = \begin{bmatrix} a_{k1} & a_{k2} & ... & a_{kn} \end{bmatrix}^T,$$

and a couple of labour coefficients matrices

$$\mathbf{L}_c = [l_{cir}]_{i=1,..,m;r=1,..,s}, \quad \mathbf{L}_k = [l_{kjr}]_{j=1,..,n;r=1,..,s}.$$

[4]However, I have no reason to expect it does.

Without loss of generality, it is assumed that two processes that produce the same good differ at least in one entry. Moreover, all entries in the above vectors and matrices are assumed to be non-negative.

Let $k(t) \geq 0$ represent the stock of capital at a given time $t \geq 0$, and

$$\mathbf{x}_k(t) = \begin{bmatrix} x_{k1}(t) & x_{k2}(t) & ... & x_{kn}(t) \end{bmatrix}^T,$$

$$\mathbf{x}_c(t) = \begin{bmatrix} x_{c1}(t) & x_{c2}(t) & ... & x_{cm}(t) \end{bmatrix}^T,$$

be the intensities of activation of the production processes at that time. Assuming that the flow of new capital is accumulated, that capital decays at a constant rate $\delta > 0$, and that the initial state of the system is $k_0 \geq 0$, then the state equation is given by the differential equation

$$\begin{cases} \dot{k}(t) = \mathbf{x}_k(t)^T \mathbf{e} - \delta k(t), & t \geq 0 \\ k(0) = k_0. \end{cases} \tag{1}$$

Assume that the different labour flows available at every t are constant and given by the strictly positive vector $\mathbf{h} = \begin{bmatrix} h_1 & h_2 & ... & h_s \end{bmatrix}^T > \mathbf{0}$, and assume that every unit of capital good instantaneously provides one unit of production services. Under these assumptions the production is subject to the following set of constraints, holding for all $t \geq 0$,

$$\mathbf{x}_c(t)^T \mathbf{L}_c + \mathbf{x}_k(t)^T \mathbf{L}_k \leq \mathbf{h}^T, \tag{2}$$

$$\mathbf{x}_c(t)^T \mathbf{a}_c + \mathbf{x}_k(t)^T \mathbf{a}_k \leq k(t), \tag{3}$$

$$\mathbf{x}_c(t) \geq \mathbf{0}, \mathbf{x}_k(t) \geq \mathbf{0}. \tag{4}$$

Let the planner's instantaneous utility be given by the amount of consumption good produced at a given time t and assume that the rate of interest (or discount) is the constant $r \geq 0$. Then the planner's problem is maximizing

$$J(\mathbf{x}_c(t), \mathbf{x}_k(t)) = \int_0^{+\infty} e^{-rt} \mathbf{x}_c(t)^T \mathbf{e} \, \mathrm{d}t \tag{5}$$

over the set of admissible controls

$$\mathcal{U}(k_0) = \{(\mathbf{x}_c(t), \mathbf{x}_k(t)) \in L^1_{loc}(0, +\infty; \mathbb{R}^{n+m}_+) \ : \ (1) - (4) \text{ hold at all } t \geq 0\}.$$

To simplify the analysis of the special features of the Hamiltonians of the problem at hand, let us make three more specific assumptions about the technology:

(H1) $\mathbf{a}_c > \mathbf{0}$ and $\mathbf{a}_k > \mathbf{0}$;

(H2) the set $\mathcal{A} = \{(\mathbf{x}_c, \mathbf{x}_k) \in \mathbb{R}^{n+m}_+ : \text{ (2) holds}\}$ is bounded;

(H3) $\exists j \in \{1, 2, ..., n\} : \delta \mathbf{e}_j^T \mathbf{a}_k < 1$.

Assumption (H1) and (H2) make respectively capital and the primary factors essential for production of the two goods. Assumption (H2), in particular, precludes unbounded growth. Assumption (H3), on the other hand, states that when no primary resource constraint is binding (i.e., when the capital stock is positive but sufficiently close to zero), then the economy can grow at a positive rate. The current value pre-Hamiltonian associated to the problem is

$$h(k, v_k, \mathbf{x}_c, \mathbf{x}_k) = \mathbf{x}_c\mathbf{e} + (\mathbf{x}_k\mathbf{e} - \delta k)v_k,$$

where v_k is the price of the capital good, while the current value Hamiltonian is

$$\begin{aligned} &H(k, v_k) = -\delta k v_k \\ &+\sup\{\mathbf{x}_c\mathbf{e}+\mathbf{x}_k\mathbf{e}v_k : (\mathbf{x}_c, \mathbf{x}_k) \in \mathbb{R}_+^{n+m},\ \mathbf{x}_c^T\mathbf{L}_c+\mathbf{x}_k^T\mathbf{L}_k \le \mathbf{h}^T,\ \mathbf{x}_c^T\mathbf{a}_c+\mathbf{x}_k^T\mathbf{a}_k \le k\}. \end{aligned} \tag{6}$$

The maximization process through which the Hamiltonian is computed is therefore equivalent to solving the following linear programming problem

$$\max[\mathbf{x}_c^T\mathbf{e} + \mathbf{x}_k^T\mathbf{e}v_k] \tag{7}$$

subject to

$$\mathbf{x}_c^T\mathbf{L}_c + \mathbf{x}_k^T\mathbf{L}_k \le \mathbf{h}^T, \tag{8}$$

$$\mathbf{x}_c^T\mathbf{a}_c + \mathbf{x}_k^T\mathbf{a}_k \le k, \tag{9}$$

$$\mathbf{x}_c \ge \mathbf{0}, \mathbf{x}_k \ge \mathbf{0}. \tag{10}$$

Under assumption (H1) (or (H2)) the feasible region is bounded and the maximum exists. The corresponding dual problem is

$$\min[kq + \mathbf{h}^T\mathbf{w}] \tag{11}$$

subject to

$$\mathbf{e} \le \mathbf{a}_c q + \mathbf{L}_c\mathbf{w} \tag{12}$$

$$\mathbf{e}v_k \le \mathbf{a}_k q + \mathbf{L}_k\mathbf{w} \tag{13}$$

$$q \ge 0, \mathbf{w} \ge \mathbf{0}, \tag{14}$$

where $q \in \mathbb{R}_+$ and $\mathbf{w} \in \mathbb{R}_+^s$ are dual control variables having the economic meaning, respectively, of the rental rate of the capital good and the wage rates. Since the primal has an optimal solution, the dual has an optimal solution too (see e.g. Franklin, 1980). For any given pair of stock of capital and capital price at time t, $k(t)$ and $v_k(t)$, the short-run competitive factor prices are the solutions of this dual problem.

A *modified golden rule* (or simply a *golden rule* if the rate of interest is zero) is a solution of the above primal and dual linear programs that satisfies the additional stationary conditions:

$$q = (\delta + r)v_k, \tag{15}$$

$$\mathbf{x}_k^T \mathbf{e} = \delta k. \tag{16}$$

It is known that a primal component of a modified golden rule is a stationary solution of the planner problem (5), and that, vice versa, any stationary optimal solution of the planner problem (5) can be supported by a stationary price system. Moreover, it is also known that if the golden rule capital stock is unique, then the primal component of a golden rule is an optimal overtaking solution of the planner problem (5) (see Lemma 3.3 and Example 4.1 of Leizarowitz, 1985, see also Example 4.4 of Carlson *et al.* , 1991). We can now use equation (16) to substitute for k in inequality (9) and equation (15) to substitute for v_k in inequality (13). This reduces the problem of identifying the modified golden rules (and the golden rules) to finding the solutions of the following *linear complementarity problem*:

$$\begin{bmatrix} \mathbf{x}_c^T & \mathbf{x}_k^T \end{bmatrix} \begin{bmatrix} \mathbf{L}_c & \mathbf{a}_c \\ \mathbf{L}_k & \mathbf{a}_k - \delta^{-1}\mathbf{e} \end{bmatrix} \leq \begin{bmatrix} \mathbf{h}^T & 0 \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} \mathbf{x}_c^T & \mathbf{x}_k^T \end{bmatrix} \begin{bmatrix} \mathbf{L}_c & \mathbf{a}_c \\ \mathbf{L}_k & \mathbf{a}_k - \delta^{-1}\mathbf{e} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ q \end{bmatrix} = \mathbf{h}^T \mathbf{w} \tag{18}$$

$$\begin{bmatrix} \mathbf{e} \\ \mathbf{0} \end{bmatrix} \leq \begin{bmatrix} \mathbf{L}_c & \mathbf{a}_c \\ \mathbf{L}_k & \mathbf{a}_k - (\delta + r)^{-1}\mathbf{e} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ q \end{bmatrix} \tag{19}$$

$$\mathbf{x}_c^T \mathbf{e} = \begin{bmatrix} \mathbf{x}_c^T & \mathbf{x}_k^T \end{bmatrix} \begin{bmatrix} \mathbf{L}_c & \mathbf{a}_c \\ \mathbf{L}_k & \mathbf{a}_k - (\delta + r)^{-1}\mathbf{e} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ q \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} \mathbf{x}_c^T & \mathbf{x}_k^T \end{bmatrix} \geq \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T \end{bmatrix}, \begin{bmatrix} \mathbf{w} \\ q \end{bmatrix} \geq \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix}. \tag{21}$$

Define $R = inf\{r \in \mathbb{R} \ : \ \mathbf{a}_k - (\delta + r)^{-1}\mathbf{e} > \mathbf{0}\}$ (observe $R > 0$ by assumption (H3)). If $r \geq R$, then a null vector of wage rates and any rental rate q such that $\mathbf{e} \leq \mathbf{a}_c q$ constitute a set of supporting factor prices for a candidate solution of the complementary problem (17)-(21) in which $\mathbf{x}_k = \mathbf{0}$ and $\mathbf{x}_c = \mathbf{0}$. This proves that the complementary problem (17)-(21) has a solution for $r \geq R$. Since in addition it can be proved that non-trivial solutions to the complementary problem (17)-(21) exist when $R > r \geq 0$, we have the following result.

Proposition 2.1 *Assume Hypotheses (H1)-(H3). Then a solution for the linear complementarity problem (17)-(21) exists for any non-negative rate of interest.*

Proof. see the Appendix. □

Remark 2.2 When $r = 0$, the gap between the matrix in (17)/(18) and the matrix in (19)/(20) vanishes, and the linear complementarity problem collapses to the standard pair of dual linear programs that characterize the golden rule: the primal maximizes steady state consumption, while the dual minimizes steady state rents for the primary factors. Note that under the expansibility assumption (H3) the golden rule consumption flow and the golden rule capital stock are both strictly positive.

Let $\mathbf{X}_c(r)$ and $\mathbf{X}_k(r)$ now be the (non-empty) sets of the intensity levels vectors $\mathbf{x}_c$ and $\mathbf{x}_k$ belonging to a solution of (17)-(21) at a given $r \geq 0$. Define the two following correspondences: the steady state capital stock $K^s(r) = \{k : k = \delta^{-1}\mathbf{x}_k^T\mathbf{e},\ \mathbf{x}_k \in \mathbf{X}_k(r)\}$, and the steady state consumption flow $C^s(r) = \{c : c = \mathbf{x}_c^T\mathbf{e},\ \mathbf{x}_c \in \mathbf{X}_c(r)\}$, and the following four step functions: $K_+(r) = \max_{k\in K^s(r)} k$, $K_-(r) = \min_{k\in K^s(r)} k$, $C_+(r) = \max_{c\in C^s(r)} c$ and $C_-(r) = \min_{c\in C^s(r)} c$. To begin with, it is useful to know how the comparative statics is shaped by the set of inequalities (17)-(21). We have the following result.

Lemma 2.3 *Let* $\begin{bmatrix}\mathbf{x}_c^T(r_1) & \mathbf{x}_k^T(r_1)\end{bmatrix}$, $\begin{bmatrix}\mathbf{w}^T(r_1) & q(r_1)\end{bmatrix}^T$ *and* $\begin{bmatrix}\mathbf{x}_c^T(r_2) & \mathbf{x}_k^T(r_2)\end{bmatrix}$, $\begin{bmatrix}\mathbf{w}^T(r_2) & q(r_2)\end{bmatrix}^T$ *be any two (modified) golden rules at given levels of the rate of interest* r_1 *and* r_2*, where without loss of generality we can set* $r_1 \leq r_2$*. Define* $c^s(r_1) = \mathbf{x}_c^T(r_1)\mathbf{e} \in C^s(r_1)$, $k^s(r_1) = \delta\mathbf{x}_k^T(r_1)\mathbf{e} \in K^s(r_1)$, $c^s(r_2) = \mathbf{x}_c^T(r_2)\mathbf{e} \in C^s(r_2)$*, and* $k^s(r_2) = \delta\mathbf{x}_k^T(r_2)\mathbf{e} \in K^s(r_2)$*. Then*

$$c^s(r_2) - c^s(r_1) \leq \frac{r_1}{r_1+\delta}q(r_1)[k^s(r_2) - k^s(r_1)], \tag{22}$$

and

$$c^s(r_1) - c^s(r_2) \leq \frac{r_2}{r_2+\delta}q(r_2)[k^s(r_1) - k^s(r_2)]. \tag{23}$$

Moreover,

$$0 \leq \mathbf{h}^T\mathbf{w}(r_2) - \mathbf{h}^T\mathbf{w}(r_1) + [\frac{r_2}{r_2+\delta}q(r_2) - \frac{r_1}{r_1+\delta}q(r_1)]k^s(r_1). \tag{24}$$

Proof. Noting that

$$\begin{bmatrix}\mathbf{L}_c & \mathbf{a}_c \\ \mathbf{L}_k & \mathbf{a}_k - (\delta+r)^{-1}\mathbf{e}\end{bmatrix} = \begin{bmatrix}\mathbf{L}_c & \mathbf{a}_c \\ \mathbf{L}_k & \mathbf{a}_k - \delta^{-1}\mathbf{e}\end{bmatrix} + \begin{bmatrix}\mathbf{0} & \mathbf{0} \\ \mathbf{0} & r\delta^{-1}(\delta+r)^{-1}\mathbf{e}\end{bmatrix},$$

we can immediately use equations (18) and (20) to obtain:

$$c^s(r_1) = \mathbf{h}^T\mathbf{w}(r_1) + \frac{r_1}{r_1+\delta}q(r_1)k^s(r_1) \tag{25}$$

and

$$c^s(r_2) = \mathbf{h}^T\mathbf{w}(r_2) + \frac{r_2}{r_2+\delta}q(r_2)k^s(r_2). \tag{26}$$

Inequalities (17) and (19), on the other hand, imply that when the primal component of a modified golden rule is evaluated at the prices of the other modified golden rule no gains in value can arise. In particular, multiplying (17) and (19) by the two sets of prices and quantities, we get:

$$c^s(r_1) \leq \mathbf{h}^T\mathbf{w}(r_2) + \frac{r_2}{r_2+\delta}q(r_2)k^s(r_1) \tag{27}$$

and

$$c^s(r_2) \leq \mathbf{h}^T\mathbf{w}(r_1) + \frac{r_1}{r_1+\delta}q(r_1)k^s(r_2). \tag{28}$$

Then subtracting (25) from (28), (26) from (27), and (25) from (27) gives the claims. □

Remark 2.4 Obviously, the economic meaning of equations (25) and (26) is that in any modified golden rule the value of the net product equals the sum of wages and profits (remember that in steady state we have $q = (\delta + r)v_k$).

Remark 2.5 Lemma 2.3 establishes that the Burmeister-like inequalities (22) and (23) hold in our multi-primary factor set-up. See equation (A14) in Burmeister (1974, p. 453) for discrete technologies with joint technologies and heterogeneous capital goods, but a single primary factor, and Theorem 7 and Theorem 9 in Burmeister Dobell (1970, p. 286 and p. 293, respectively) for non-joint production neoclassical technologies, still with homogeneous labour as the single primary factor. See also Burmeister (1976) for a similar result. Note that inequality (22) implies that an increase in the rate of interest can raise steady-state consumption only if a positive "real Wicksell effect" (i.e., $\Delta k/\Delta r > 0$) occurs. When the price of capital is positive, inequality (23) ensures that a positive "real Wicksell effect" is also sufficient for a rise in steady-state consumption.

As a corollary of Lemma 2.3, we also establish a link between the sign of the "real Wicksell effect", the sign of the "price Wicksell effect" (i. e., $\Delta v_k/\Delta r$) and the sign of the effect on steady state wages of an increase in the rate of interest.

Corollary 2.6 *Consider the modified golden rules given in Lemma 2.3. Then a positive "real Wicksell effect" cannot occur if*

(i) $\frac{\Delta \mathbf{h}^T \mathbf{w}}{\Delta r} < 0$, *or*

(ii) the "price Wicksell effect" is positive, or

(iii) the "price Wicksell effect" is non-negative and the price of capital is positive.

Proof. Let $\Delta r = r_2 - r_1 > 0$. To prove the first claim, suppose $\mathbf{h}^T \mathbf{w}(r_2) - \mathbf{h}^T \mathbf{w}(r_1) < 0$. Then $k(r_1) > 0$, because at $r = r_1$ at least a wage rate is positive and, hence, some kind of labour is fully employed. Thus Assumption (H1) implies that the capital stock is positive. From inequality (24) we therefore have $\frac{r_2}{r_2+\delta} q(r_2) - \frac{r_1}{r_1+\delta} q(r_1) > 0$. Since adding (22) and (23) gives

$$0 \leq -[\frac{r_2}{r_2 + \delta} q(r_2) - \frac{r_1}{r_1 + \delta} q(r_1)][k^s(r_2) - k^s(r_1)], \tag{29}$$

we then conclude that $k^s(r_2) - k^s(r_1) > 0$ is impossible. Suppose now $k^s(r_2) - k^s(r_1) > 0$. Then (29) gives $\frac{r_2}{r_2+\delta} q(r_2) - \frac{r_1}{r_1+\delta} q(r_1) \leq 0$ and hence

$$\frac{q(r_2)}{r_2 + \delta} \leq \frac{r_1}{r_2} \frac{q(r_1)}{r_1 + \delta} \leq \frac{q(r_1)}{r_1 + \delta},$$

with the second inequality strict if $q(r_1) > 0$. This proves our last two claims and concludes the proof. $\square$

As is well known, when there is a single primary factor, the hypothesis in point (i) of Corollary (2.6) is satisfied for all $0 \leq r_1 < r_2 \leq R$.[5] On the other hand, the hypothesis in point (iii) of Corollary (2.6) is obviously satisfied if the capital good and the consumption good share the same technology. These facts imply the following result:

Proposition 2.7 *Assume Hypotheses (H1)-(H3) and at least one of the following conditions holds: (i)* $\mathbf{L}_k = \mathbf{L}_c \equiv \mathbf{L}$ *and* $\mathbf{a}_k = \mathbf{a}_c \equiv \mathbf{a}$*, (ii)* $s = 1$*. Then (a)* $K_+(r)$*,* $K_-(r)$*,* $C_+(r)$*,* $C_-(r)$ *are decreasing step functions, (b)* $K_+(r) \neq K_-(r)$ *or* $C_+(r) \neq C_-(r)$ *(or both) only for a finite set of values of the interest rate* r*, (c)* $K_+(r) \neq K_-(r)$ *implies* $K^s(r) = [K_-(r), K_+(r)]$ *and* $C_+(r) \neq C_-(r)$ *implies* $C^s(r) = [C_-(r), C_+(r)]$.

Proof. We give here a self-contained proof of these classical results only for the case in which condition (i) holds, where, after reducing the model to an aggregate model, the argument can proceed along the lines usually used for the one-sector Ramsey-Solow model. Note that if hypothesis (i) is valid the marginal rate of transformation between the consumption good and the capital good is 1. Thus across all the steady states (where both goods are produced) $v_k = 1$ holds. Then linear parametric programming can be used to construct the aggregate "Ricardian" production function

$$y(k) = \max[\mathbf{y}^T \mathbf{e}] \tag{30}$$

subject to

$$\mathbf{y}^T \mathbf{L} \leq \mathbf{h}^T, \tag{31}$$

$$\mathbf{y}^T \mathbf{a} \leq k, \tag{32}$$

$$\mathbf{y} \geq \mathbf{0}, \tag{33}$$

where $\mathbf{y} = \mathbf{x}_c + \mathbf{x}_k$. By the linear parametric programming theory, the function $y(k)$ is piecewise linear and concave (see e.g. Franklin, 1980, pp.69-72); by assumption (H1), $y(0) = 0$, and by assumption (H3), $y'_+(0) > \delta$. Since duality implies $r + \delta = q \in \partial y(k)$, then $K^s(r)$ is a convex correspondence nonincreasing in r. The result for the steady state consumption follows by noting that $k \in K^s(r) \iff y(k) - \delta k \in C^s(r)$. The proof if condition (ii) holds is quite standard and is omitted here. For the sake of completeness it is given in the Appendix. □

The next section shows that the "neoclassical" properties in Proposition 2.7 do not carry over to the general two-sector one capital good multi-primary-factor model even if joint production is excluded.

[5] Provided there is no joint production, with homogeneous labour, the hypothesis holds even if capital is heterogeneous. However, as became clear in the course of the capital controversy, the absence of positive real Wicksell effects is implied only if there only one capital good. In the literature reference is often made to the decreasing *wage-profit frontier*, but other expressions are also used. For example, in Samuelson & Etula, 2006 p. 331, the fact that $\frac{\Delta w}{\Delta r} < 0$ is dubbed "Ricardo's fundamental inverse tradeoff between profit and wage rate".

3 An example

A minimal example to show that the linear complementary problem (17)-(21) can have multiple solutions even if the price of capital is everywhere positive is provided in this section. In the example, only two primary factors of production (two qualities of labour) are used in both sectors and only a single process for the production of the capital good and two processes for the production of the consumption good are available. The labour matrices and the capital vectors of the coefficients are as follows:

$$\mathbf{L}_c = \begin{bmatrix} \frac{95}{100} & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{l}_k = \begin{bmatrix} \frac{1}{100} & \frac{11}{10} \end{bmatrix}$$

$$\mathbf{a}_c = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \end{bmatrix}^T$$

$$a_k = \frac{1}{4},$$

and the rate of decay of capital δ is assumed to be 1. Moreover, the system is endowed with one unit of labour of type 1 and two units of labour of type 2. Hence we have $\mathbf{h} = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$, and thus the linear complementary problem (17)-(21) in this specific example takes the form:

$$\begin{bmatrix} x_{c1} & x_{c2} & x_k \end{bmatrix} \begin{bmatrix} \frac{95}{100} & 1 & \frac{1}{2} \\ 1 & 2 & \frac{1}{4} \\ \frac{1}{100} & \frac{11}{10} & \frac{1}{4} - 1 \end{bmatrix} \leq \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \tag{34}$$

$$\begin{bmatrix} x_{c1} & x_{c2} & x_k \end{bmatrix} \begin{bmatrix} \frac{95}{100} & 1 & \frac{1}{2} \\ 1 & 2 & \frac{1}{4} \\ \frac{1}{100} & \frac{11}{10} & \frac{1}{4} - 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ q \end{bmatrix} = w_1 + 2w_2 \tag{35}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \leq \begin{bmatrix} \frac{95}{100} & 1 & \frac{1}{2} \\ 1 & 2 & \frac{1}{4} \\ \frac{1}{100} & \frac{11}{10} & \frac{1}{4} - \frac{1}{1+r} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ q \end{bmatrix} \tag{36}$$

$$x_{c1} + x_{c2} = \begin{bmatrix} x_{c1} & x_{c2} & x_k \end{bmatrix} \begin{bmatrix} \frac{95}{100} & 1 & \frac{1}{2} \\ 1 & 2 & \frac{1}{4} \\ \frac{1}{100} & \frac{11}{10} & \frac{1}{4} - \frac{1}{1+r} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ q \end{bmatrix} \tag{37}$$

$$\begin{bmatrix} x_{c1} & x_{c2} & x_k \end{bmatrix} \geq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ q \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{38}$$

Note that $R = 3$. For $0 \leq r < 3$ both goods are produced. Thus either one of the processes producing the consumption good is not operated or all three processes are activated. In the latter case, system (36) must hold with equality, which means, after substituting for q using the last equation in (36) in the first two equations, that the system

$$[\frac{95}{100} + \frac{1+r}{50(3-r)}]w_1 + [1 + \frac{11(1+r)}{5(3-r)}]w_2 = 1 \tag{39}$$

$$[1 + \frac{1+r}{100(3-r)}]w_1 + [2 + \frac{11(1+r)}{10(3-r)}]w_2 = 1 \tag{40}$$

must hold. A straightforward calculation shows that a non-negative solution for this system exists if and only if $\frac{19}{21} \leq r \leq \frac{7}{3}$. At $r = \frac{19}{21}$, $w_1 = 0$ and $w_2 = \frac{1}{3}$, while at $r = \frac{7}{3}$, $w_1 = \frac{20}{21}$ and $w_2 = 0$. In between, both wage rates are positive. Observe that total wages equal $\frac{2}{3}$ for $r = \frac{19}{21}$ and $\frac{20}{21}$ for $r = \frac{7}{3}$. Hence $\frac{\Delta \mathbf{h}^T \mathbf{w}}{\Delta r}$ cannot be everywhere negative. Since solving system (34) with equality gives

$$\begin{bmatrix} x_{c1} & x_{c2} & x_k \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{95}{100} & 1 & \frac{1}{2} \\ 1 & 2 & \frac{1}{4} \\ \frac{1}{100} & \frac{11}{10} & \frac{1}{4} - 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{24}{35} & \frac{12}{35} & \frac{20}{35} \end{bmatrix}, \tag{41}$$

then $\frac{36}{35} \in C^s(r)$ and $\frac{20}{35} \in K^s(r)$ for $\frac{19}{21} \leq r \leq \frac{7}{3}$. Moreover, no other solution in which all the three processes are operated exists for $\frac{19}{21} < r < \frac{7}{3}$.

Consider next the case in which a single process is operated in the consumption sector. In this case, system (34) cannot hold with equality, implying that unemployment of one type of labour arises. Since the first process uses type 1 labour more intensively than the second process, operating the first process induces unemployment of type 2 labour, while activating the second process leads to less than full employment for the labour of kind 1. In particular, setting $x_{c2} = 0$ and assuming that the first and last inequality in system (34) hold with equality leads to the system

$$\frac{95}{100}x_{c1} + \frac{1}{100}x_k = 1 \tag{42}$$

$$x_{c1} + \frac{11}{10}x_k \leq 2 \tag{43}$$

$$\frac{1}{2}x_{c1} - \frac{3}{4}x_k = 0, \tag{44}$$

whose solution is given by $x_{c1} = \frac{300}{287}$, $x_k = \frac{200}{287}$. Setting instead $x_{c1} = 0$ and assuming that the second and the last inequality in system (34) hold with equality leads to the system

$$x_{c2} + \frac{1}{100}x_k \leq 1 \tag{45}$$

$$2x_{c2} + \frac{11}{10}x_k = 2 \tag{46}$$

$$\frac{1}{4}x_{c2} - \frac{3}{4}x_k = 0, \tag{47}$$

that has the solution $x_{c2} = \frac{60}{71}$, $x_k = \frac{20}{71}$.

Note that $\frac{300}{287}$ is the golden rule consumption flow (i. e. $\frac{300}{287}$ is the only element in $C^s(0)$). This implies that there is a price support for $\begin{bmatrix} x_{c1} & x_{c2} & x_k \end{bmatrix} = \begin{bmatrix} \frac{300}{287} & 0 & \frac{200}{287} \end{bmatrix}$ if the rate of interest is literally zero or close to zero. To find explicitly the supporting prices and the interval of existence of these prices, set $w_2 = 0$ in system (36) and assume the first and the last inequality hold with equality. This leads to

$$\frac{95}{100}w_1 + \frac{1}{2}q = 1 \tag{48}$$

$$w_1 + \frac{1}{4}q \geq 1 \tag{49}$$

$$\frac{1}{100}w_1 + [\frac{1}{4} - \frac{1}{1+r}]q = 0, \tag{50}$$

which, for $0 \leq r \leq \frac{7}{3}$, has the solution $w_1 = [\frac{95}{100} + \frac{1+r}{50(3-r)}]^{-1} > 0$, $q = \frac{1}{100}[\frac{95}{100} + \frac{1+r}{50(3-r)}]^{-1}[\frac{1}{1+r} - \frac{1}{4}]^{-1} > 0$. Note that at $r = \frac{7}{3}$ inequality (49) holds with equality. In the same way, setting $w_1 = 0$ and verifying that for $\frac{19}{21} \leq r \leq 3$ the system

$$w_2 + \frac{1}{2}q \geq 1 \tag{51}$$

$$2w_2 + \frac{1}{4}q = 1 \tag{52}$$

$$\frac{11}{10}w_2 + [\frac{1}{4} - \frac{1}{1+r}]q = 0, \tag{53}$$

has the solution $w_2 = \frac{10(3-r)}{71-9r} \geq 0$, $q = 4 - \frac{80(3-r)}{71-9r} > 0$ proves that $\begin{bmatrix} x_{c1} & x_{c2} & x_k \end{bmatrix} = \begin{bmatrix} 0 & \frac{60}{71} & \frac{20}{71} \end{bmatrix}$ has a support for $\frac{19}{21} \leq r \leq 3$. For $r = 3$ both wage rates are zero, such that both kinds of labour can be unemployed. This implies that at $r = 3$ there is a supporting price for any vector $\begin{bmatrix} x_{c1} & x_{c2} & x_k \end{bmatrix} = \begin{bmatrix} 0 & \theta\frac{60}{71} & \theta\frac{20}{71} \end{bmatrix}$, where $0 \leq \theta \leq 1$. The above argument therefore establishes three further facts. First, $\frac{300}{287} \in C^s(r)$ and $\frac{200}{287} \in K^s(r)$ for $0 \leq r \leq \frac{7}{3}$, and $\frac{60}{71} \in C^s(r)$ and $\frac{20}{71} \in K^s(r)$ for $\frac{19}{21} \leq r \leq 3$. Second, no other solution in which two processes are operated exists for $0 \leq r < 3$. Third, $[0 \quad \frac{60}{71}] \subseteq C^s(3)$ and $[0 \quad \frac{20}{71}] \subseteq K^s(3)$.

Finally, consider the critical cases $r = \frac{19}{21}$, where the first kind of labour needs not be fully employed, and $r = \frac{7}{3}$, where instead some labour of type 2 can be unemployed. Since the price system at $r = \frac{19}{21}$ supports any convex combinations of the two intensity vectors $\begin{bmatrix} \frac{24}{35} & \frac{12}{35} & \frac{20}{35} \end{bmatrix}$ and $\begin{bmatrix} 0 & \frac{60}{71} & \frac{20}{71} \end{bmatrix}$, then $[\frac{60}{71} \quad \frac{36}{35}] \subseteq C^s(\frac{19}{21})$ and $[\frac{20}{71} \quad \frac{20}{35}] \subseteq K^s(\frac{19}{21})$. Analogously, the price system at $r = \frac{7}{3}$ supports any convex combinations of the two intensity vectors $\begin{bmatrix} \frac{24}{35} & \frac{12}{35} & \frac{20}{35} \end{bmatrix}$

and $\left[\frac{300}{287} \quad 0 \quad \frac{200}{287}\right]$. Thus, $\left[\frac{36}{35} \quad \frac{300}{287}\right] \subseteq C^s(\frac{7}{3})$ and $\left[\frac{20}{35} \quad \frac{200}{287}\right] \subseteq K^s(\frac{7}{3})$. This concludes the construction of the $C^s(r)$ and $K^s(r)$ correspondences for the current example.

The results for $K^s(r)$ are plotted in Figure 1. The graph of the $C^s(r)$ correspondence is similar and is depicted in Figure 2. For $\frac{19}{21} < r < \frac{7}{3}$ there are three steady states. For this range of values of the interest rate, positive real Wicksell effects that imply $\frac{\Delta c}{\Delta r} > 0$ are clearly possible. However, by a straightforward analysis of the Hamiltonian dynamics, which is not developed here, it can be proved that the equilibria are alternately stable and unstable (see e.g., Liviatan & Samuelson, 1969; Freni, 1997). Thus the instability of the intermediate equilibrium (those on the segment DC, where there is an increasing step in the graph of the $K^s(r)$ correspondence) precludes comparative statics perverse effects for stable equilibria.[6] Nevertheless, in this example, the long-run wages distribution is history-dependent. Interestingly, the results also show that the equilibria with a wide wage gap are the stable ones.

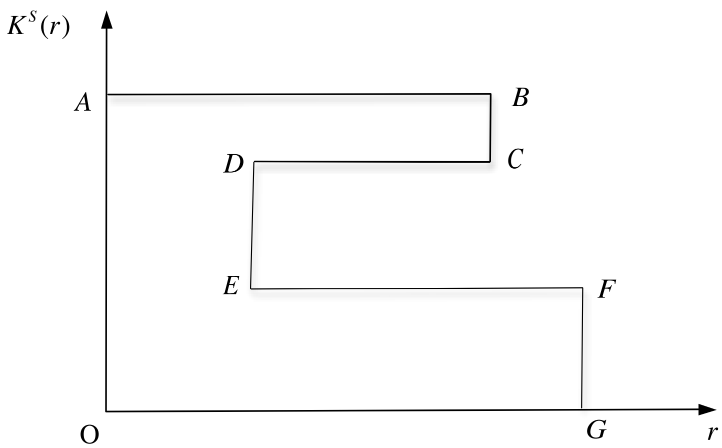

Figure 1: The graph of the steady state $K^s(r)$ correspondence. $OG = 3$ gives the maximum rate of interest and $OA = \frac{200}{287}$ is the golden rule stock level. $AB = \frac{7}{3}$ and $DC = \frac{7}{3} - \frac{19}{21}$. Two relevant modified golden rules stock levels are given by $FG = \frac{20}{71}$ and $FG + DE = \frac{36}{35}$.

[6] For the workings of Samuelson's Correspondence Principle with a scalar state variable see Burmeister & Long (1977).

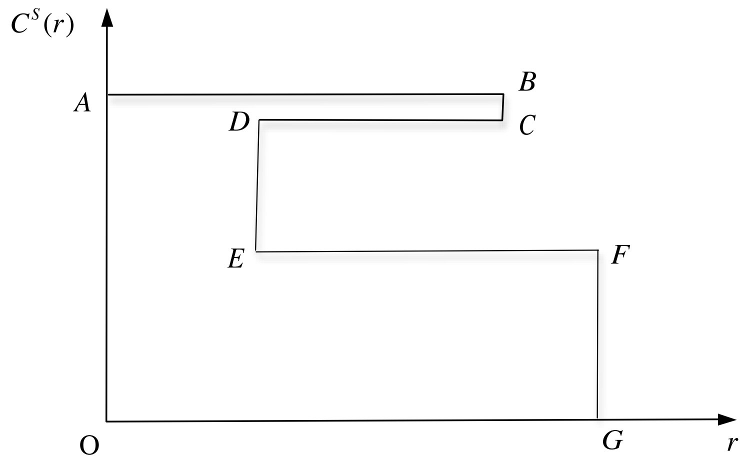

Figure 2: The graph of the steady state $C^s(r)$ correspondence. $OA = \frac{300}{287}$ is the golden rule consumption level. Two relevant modified golden rule consumption flows are given by $FG = \frac{60}{71}$ and $FG + DE = \frac{20}{35}$.

4 Concluding remarks

This paper shows non-uniqueness of the steady state for Bruno's (1967) one capital good two-sector growth model with a discrete technology when there are many primary factors of production. Of course, it is natural to expect the same problem to arise in Uzawa's (1964) two-sector neoclassical model with heterogeneous labour. The exploration of this issue is left for future investigation. Clark's smooth version of the model, often slightly modified to include a non-linear utility function, has also been used as a dynamic Heckscher-Ohlin framework for the analysis of convergence among open economies (see e.g. Atkeson & Kehoe, 2000 or Stiglitz, 1970, for an early contribution). At times, multiple primary factors have been included in the model (e.g. Nishimura *et al.* 2006, Guilló & Perez-Sebastian, 2015), but the full extended model has not yet been worked out.

The model can be reinterpreted as a two-agricultural-good Ricardian model with multiple qualities of land (Samuelson, 1959; Pasinetti, 1960). In this case, if capitalists require a positive rate of profits to carry on a stationary stock, then our results prove that the uniqueness of the stationary state is not guaranteed.

While the problem of uniqueness of the steady state in growth models has been explored in some generality in the case of a single primary factor (see e.g., Brock, 1973; Brock & Burmeister, 1976; Burmeister, 1981),[7] not much is

[7] All these results apply to smooth economies.

known about what kind of economically relevant conditions lead to uniqueness in multi-primary-factor models. This too seems an interesting question for future work.

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A Proof of Proposition 2.1

Proof. We already know that the trivial steady state can be supported by a price system if $r \geq R$. For the case in which $r < R$ we show that, under our assumptions, the key hypothesis of the Complementary Construction Theorem in Dantzig & Manne (1974) is satisfied.

Define $\mathbf{C}^T = \begin{bmatrix} \mathbf{L}_c & \mathbf{a}_c \\ \mathbf{L}_k & \mathbf{a}_k - \delta^{-1}\mathbf{e} \end{bmatrix}$, $\mathbf{D}^T = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r\delta^{-1}(\delta + r)^{-1}\mathbf{e} \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} \mathbf{x}_c \\ \mathbf{x}_k \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} w \\ \mathbf{q} \end{bmatrix}$, $\mathbf{l} = \begin{bmatrix} \mathbf{h} \\ 0 \end{bmatrix}$ and $-\mathbf{f} = \begin{bmatrix} -\mathbf{e} \\ \mathbf{0} \end{bmatrix}$ and rewrite the linear complementary problem (17)-(21) as follows:

$$\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \equiv \begin{bmatrix} \mathbf{0} & -\mathbf{C} \\ \mathbf{C}^T + \mathbf{D}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{x} \end{bmatrix} + \begin{bmatrix} \mathbf{l} \\ -\mathbf{f} \end{bmatrix} \geq \mathbf{0}$$

$$\begin{bmatrix} \mathbf{v} \\ \mathbf{x} \end{bmatrix} \geq \mathbf{0}$$

$$\begin{bmatrix} \mathbf{v}^T & \mathbf{x}^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = 0.$$

Dantzig & Manne (1974) proved that the Lemke algorithm leads to a solution of this problem, provided the sets of optimal solutions of the two following linear programming problems are both nonempty and bounded:

$$\max \begin{bmatrix} \mathbf{e}^T & \mathbf{0}^T \end{bmatrix} \begin{bmatrix} \mathbf{x}_c \\ \mathbf{x}_k \end{bmatrix} \tag{54}$$

subject to

$$\begin{bmatrix} \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}_c \\ \mathbf{x}_k \end{bmatrix} \leq \begin{bmatrix} \mathbf{h} \\ 0 \end{bmatrix} \tag{55}$$

$$\begin{bmatrix} \mathbf{x}_c \\ \mathbf{x}_k \end{bmatrix} \geq \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}; \tag{56}$$

and

$$\min \begin{bmatrix} \mathbf{h}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ q \end{bmatrix} \tag{57}$$

subject to

$$\begin{bmatrix} \mathbf{C}^T + \mathbf{D}^T \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ q \end{bmatrix} \geq \begin{bmatrix} \mathbf{e} \\ \mathbf{0} \end{bmatrix} \tag{58}$$

$$\begin{bmatrix} \mathbf{w} \\ q \end{bmatrix} \geq \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix}. \tag{59}$$

The null vector is feasible both for problem (54)-(56) and for the dual of problem (57)-(59). Moreover, choosing a sufficiently large $m > 0$, the vector $\begin{bmatrix} m\mathbf{e} \\ 0 \end{bmatrix}$ is feasible both for problem (57)-(59) and for the dual of problem (54)-(56). Thus both the above linear programming problems have optimal solutions (see Franklin, 1980). The set of optimal solutions of the linear program (54)-(56) is bounded because of assumption (H2). For problem (57)-(59), note that since the vector of optimal wage rates $\mathbf{w}$ is obviously bounded, then also the optimal rental rate q is bounded. Note indeed that $\left[(\delta + r)^{-1}\mathbf{e} - \mathbf{a}_k\right] q \leq \mathbf{L}_k \mathbf{w}$ from inequality (19) and that the vector $\left[(\delta + r)^{-1}\mathbf{e} - \mathbf{a}_k\right]$ has at least a positive entry because $r < R$. □

B Proof of Proposition 2.7 in the case of homogeneous labour

We need some preliminary results linking the solutions of the linear complementary problem (17)-(21) to the wage-profit frontier. Suppose labour is homogeneous. Let $\mathbf{l}_c$ and $\mathbf{l}_k$ denote in this case the $m \times 1$ and $n \times 1$ labour input matrices. Observe that assumption (H2) implies that $\mathbf{l}_c$ and $\mathbf{l}_k$ are both strictly positive. Normalize to 1 the constant flow of labour and consider the family of linear programming problems

$$w^*(r) = \min w \tag{60}$$

subject to

$$\begin{bmatrix} \mathbf{e} \\ \mathbf{0} \end{bmatrix} \leq \begin{bmatrix} \mathbf{l}_c & \mathbf{a}_c \\ \mathbf{l}_k & \mathbf{a}_k - (\delta + r)^{-1}\mathbf{e} \end{bmatrix} \begin{bmatrix} w \\ q \end{bmatrix} \tag{61}$$

$$\begin{bmatrix} w \\ q \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{62}$$

Since there are vectors of factor prices with $q = 0$ and a sufficiently high wage rate that are feasible no matter how high the interest rate, then the problem has an optimal solution for any $r \geq 0$. Depending on the value of the rate of interest ($0 \leq r < R$, $r = R$, or $r > R$), we have three different kinds of solutions. Figures 3 and 4 show what the non-extra-profits region defined by (61) and (62) typically looks like in the extreme cases $r = 0$ and $r > R$. Since $(\mathbf{e}_j^T \mathbf{l}_k)^{-1}[(\delta + r)^{-1} - \mathbf{e}_j^T \mathbf{a}_k]$ gives the slope of the zero-profit line for the capital producing j-th method, increasing the interest rate with $r \geq 0$ generates a clockwise rotation of all the n capital-related lines. At $r = R$ the slope of the zero-profit line of any capital-producing process is no greater than 0, but there is at least one process for which the slope is zero. For $0 \leq r < R$, $w^*(r) > 0$ and $\Delta w^*(r)/\Delta r < 0$, while the optimal rental rate $q^*(r)$ is unique and positive, but $\Delta q^*(r)/\Delta r > 0$. The continuous decreasing function $w^*(r)$ gives the wage-profit frontier. Of course, $w^*(r) = 0$ and $\mathbf{e} \leq \mathbf{a}_c q^*(r)$ for $r \geq R$.

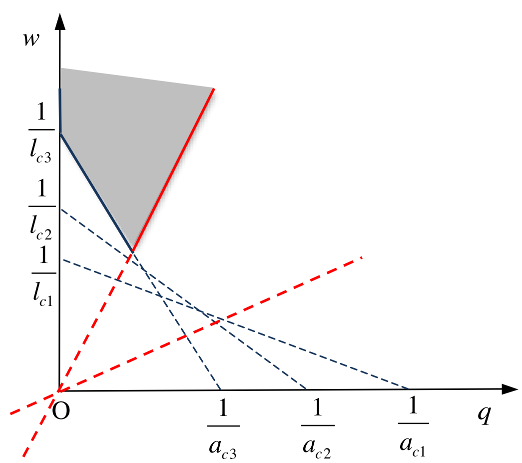

Figure 3: Case $r = 0$. In this example, there are three processes for the production of the consumption good (blue) and (at least) two processes for producing the capital good (red). The shaded region indicates where profits are non-positive and non-negativity conditions are satisfied.

The following result links the price system of the modified golden rules to the solutions of the above linear programming problem.

Lemma B.1 *Assume Hypotheses (H1)-(H3) and suppose that* $s = 1$. *Let* $\begin{bmatrix}\mathbf{x}_c^T(r) & \mathbf{x}_k^T(r)\end{bmatrix}$, $\begin{bmatrix}w(r) & q(r)\end{bmatrix}^T$ *be any (modified) golden rule at the rate of interest* $r \geq 0$. *Then* $\begin{bmatrix}w(r) & q(r)\end{bmatrix}^T$ *is a solution of the linear programming problem (60)-(62).*

Proof. Consider a solution of the complementary problem (17)-(21). If $r > R$, then clearly we have $\mathbf{x}_k(r) = \mathbf{0}$ and $\mathbf{x}_c(r) = \mathbf{0}$ (capital cannot be produced, see Figure 4, and hence nothing can be produced), so labour is unemployed

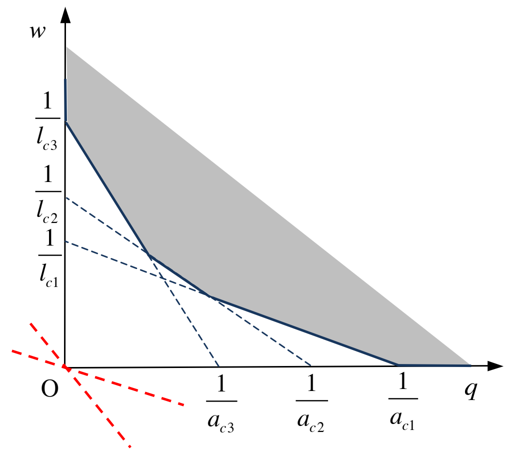

Figure 4: Case $r > R$. No process producing the capital good appears on the frontier of the feasible region. So the capital good cannot be produced without incurring losses.

and then $w(r) = 0$ and $\mathbf{e} \le \mathbf{a}_c q(r)$. The same occurs at $r = R$ if nothing is produced. If $r = R$ and something is produced, then some capital is produced and hence again $w(R) = 0$, but this time $q(R)\min_i(a_{ci}) = 1$ because $\mathbf{x}_k(R)^T(\mathbf{a}_k - \delta^{-1}\mathbf{e}) < 0$ and $\mathbf{x}_c(R)^T\mathbf{a}_c + \mathbf{x}_k(R)^T(\mathbf{a}_k - \delta^{-1}\mathbf{e}) = 0$ require the production of the consumption good. In any case, for $r \ge R$ we have $w(r) = 0$ and $\mathbf{e} \le \mathbf{a}_c q(r)$; hence $\begin{bmatrix} w(r) & q(r) \end{bmatrix}^T$ solves (60)-(62). If $0 \le r < R$, then the wage rate is positive, so there is full employment and, hence, something is necessarily produced. Some capital is then produced and so $q(r) > 0$ and, thus, $\mathbf{x}_c(r)^T\mathbf{a}_c + \mathbf{x}_k(r)^T(\mathbf{a}_k - \delta^{-1}\mathbf{e}) = 0$. Then, once again, the fact that $\mathbf{x}_k(r)^T(\mathbf{a}_k - \delta^{-1}\mathbf{e}) < 0$ implies that also the consumption good is produced. If both goods are produced, then clearly even in this case $\begin{bmatrix} w(r) & q(r) \end{bmatrix}^T$ solves (60)-(62) (see Figure 3). □

In proving this Lemma we proved *en passant* the following corollary.

Corollary B.2 *Let* $\begin{bmatrix} \mathbf{x}_c^T(r) & \mathbf{x}_k^T(r) \end{bmatrix}$, $\begin{bmatrix} w(r) & q(r) \end{bmatrix}^T$ *be any (modified) golden rule at the rate of interest* $r \ge R$. *Then* $\begin{bmatrix} \mathbf{x}_c(r)^T & \mathbf{x}_k(r)^T \end{bmatrix} \ne \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T \end{bmatrix}$ *implies* $r = R$ *and* $q(R)\min_i(a_{ci}) = 1$.

However, the non-substitution result in Lemma (B.1) also has the far-reaching implication that it allows for any $0 \le r \le R$ the reduction of the linear complementary problem (17)-(21) to the following pair of dual linear programming problems:

$$w^{**}(r) = \min w \tag{63}$$

subject to

$$\begin{bmatrix} \mathbf{e} \\ -\frac{r}{\delta(r+\delta)} q^*(r)\mathbf{e} \end{bmatrix} \leq \begin{bmatrix} \mathbf{l}_c & \mathbf{a}_c \\ \mathbf{l}_k & \mathbf{a}_k - \delta^{-1}\mathbf{e} \end{bmatrix} \begin{bmatrix} w \\ q \end{bmatrix} \tag{64}$$

$$\begin{bmatrix} w \\ q \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{65}$$

and

$$\max[\mathbf{x}_c^T \mathbf{e} - \frac{r}{\delta(r+\delta)} q^*(r) \mathbf{x}_k^T \mathbf{e}] \tag{66}$$

subject to

$$\begin{bmatrix} \mathbf{x}_c^T & \mathbf{x}_k^T \end{bmatrix} \begin{bmatrix} \mathbf{l}_c & \mathbf{a}_c \\ \mathbf{l}_k & \mathbf{a}_k - \delta^{-1}\mathbf{e} \end{bmatrix} \leq \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{67}$$

$$\begin{bmatrix} \mathbf{x}_c^T & \mathbf{x}_k^T \end{bmatrix} \geq \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T \end{bmatrix}, \tag{68}$$

where $q^*(r)$, for $r < R$, is the unique value of the modified golden rule rental rate of capital, while for $r = R$ assumes the value $\min_i (a_{ci})^{-1}$. Formally, we have the result summarized in the following Lemma.

Lemma B.3 *Assume Hypotheses (H1)-(H3) and suppose that $s = 1$. Let $0 \leq r \leq R$. Any solution of the linear complementary problem (17)-(21) solves the dual linear programming problems (63)-(65) and (66)-(68), and, conversely, any pair of solutions of the dual linear programming problems (63)-(65) and (66)-(68) solves the linear complementary problem (17)-(21).*

Proof. Once it is noted that the linear programming problems (63)-(65) and (60)-(62) have the same solution, the result follows from a straightforward verification of the fact that the conditions (17)-(21) are equivalent to optimality conditions for the linear programming problems (63)-(65) and (66)-(68). □

We can now complete the proof of Proposition 2.7.

Proof of Proposition 2.7 in case (ii) holds. For $0 \leq r \leq R$, we construct the sets $K^s(r)$ and $C^s(r)$ by solving the linear programming problem (66)-(68) in two stages. In the first stage, we pick the maximum consumption for a given capital stock no greater than the golden rule stock:

$$c^*(k) = \max \mathbf{x}_c^T \mathbf{e} \tag{69}$$

subject to

$$\begin{bmatrix} \mathbf{x}_c^T & \mathbf{x}_k^T \end{bmatrix} \begin{bmatrix} \mathbf{l}_c & \mathbf{a}_c & \mathbf{0} \\ \mathbf{l}_k & \mathbf{a}_k - \delta^{-1}\mathbf{e} & \delta^{-1}\mathbf{e} \end{bmatrix} \leq \begin{bmatrix} 1 & 0 & k \end{bmatrix} \tag{70}$$

$$\begin{bmatrix} \mathbf{x}_c^T & \mathbf{x}_k^T \end{bmatrix} \geq \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T \end{bmatrix}. \tag{71}$$

Then we identify $K^s(r)$ and $C^s(r)$ through the condition $\frac{r}{(r+\delta)} q^*(r) \in \partial c^*(k)$. Now the convexity of $K^s(r)$ and $C^s(r)$ is given by a basic result of the parametric linear programming theory that implies that $c^*(k)$ is a non-decreasing piecewise linear concave function as the one depicted in Figure 5 (see e.g. Franklin, 1980, pp.69-72), whereas the conclusion that $K_+(r)$, $C_+(r)$, $K_-(r)$ and $C_-(r)$ are step decreasing functions follows from Lemma (B.1), from which we have that $q^*(r)$ is increasing with r, and the observation that also $\frac{r}{(r+\delta)}$ is increasing with r. Recalling that Corollary (B.2) gives $K^s(r) = C^s(r) = \{0\}$ for $r > R$ ends the proof. $\square$

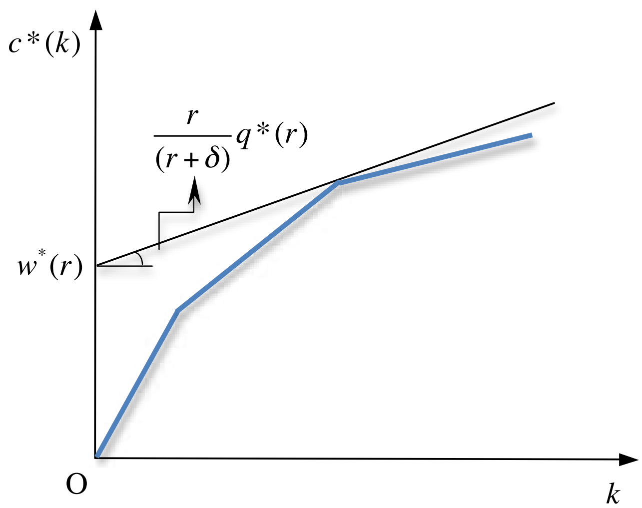

Figure 5: Steady state consumption as a function of the capital stock. Tangency with the line $w^*(r) + \frac{r}{(r+\delta)}q^*(r)k$ gives the $C^s(r)$ and $K^s(r)$ sets.